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SOME PROPERTIES OF ORDERABLE SET-FUNCTIONS

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13. ABSTRACT

A set function (not necessarily additive) on a measurable space I is called orderable if for each measurable order [Aumann, R. J. and L. S. Shapley, Values of Non-atomic games, Princeton University Press, Princeton, 1973], $\mathcal{R}$ on I there is a measure $\varphi^{\mathcal{R}}v$ on I such that for all subsets J of I that are initial segments $\varphi^{\mathcal{R}}v(J) = v(J)$. Properties like non-atomicity, nullness of sets and weak continuity are shown to be inherited from orderable set functions v to the $\varphi^{\mathcal{R}}v$'s, and vise versa. A characterization of set functions which are absolutely continuous (w.r.t. some positive measure) in the set of orderable set functions is also given.

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SOME PROPERTIES OF ORDERABLE SET-FUNCTIONS

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Introduction

A (not necessarily additive) set function v on a measurable space I is called orderable* if, for each measureable order* $\mathcal{R}$ on I there is a measure $\varphi^{\mathcal{R}}v$ on I such that for all subsets J of I that are initial segments in the order $\mathcal{R}$, we have

$$(\varphi^{\mathcal{R}}v)(J) = v(J) \quad .$$

To understand orderability intuitively, think of I as consisting of an (inhomogeneous) liquid, and of v(S) as representing some measure of the "worth" of a particular part S of I. Think of this liquid as flowing from one place to another, the drops arriving in the order $\mathcal{R}$. As it arrives, each drop of the liquid contributes to (or detracts from) the worth of that portion of the liquid already at the destination Intuitively, $(\varphi^{\mathcal{R}}v)(s)$ is the total increment contributed in this way by all the drops in a set S. Since v is in general not additive, $\varphi^{\mathcal{R}}v$ will depend strongly on $\mathcal{R}$; and in fact, it may not even exist for all $\mathcal{R}$. Orderable v are those for which it does.

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* [A-S, Section 12.]

The reader is referred to Chapter II of [A-S] for an explanation of how these notions are motivated by game-theoretic considerations.

It is the purpose of this paper to investigate the properties of the space ORD of orderable set functions. We shall establish (Section 5) that for $v \in$ ORD, certain regularity properties -- such as non-atomicity and weak continuity (defined in Section 3) -- are inherited by $\varphi^{\mathcal{R}} v$ from v; and in Section 6 we shall show that a set function is absolutely continuous* if and only if $\{\varphi^{\mathcal{R}} v\}$ is weakly sequentially compact in the space of all σ-additive measures on I.

*Ibid, Section 6.

1. Notational Conventions

The symbol $\| \|$ for a norm is used in many different senses throughout the paper; but it is never used in two different senses on the same space, so no confusion can result. When x is in an Euclidean space of finite dimension it will be noted $\vec{x}$; $\|\vec{x}\|$ will always mean the summing norm, i.e.,

$$\|\vec{x}\| = \sum_i |x_i| \quad .$$

It is important to distinguish between functions and their values. For example, if μ is a measure, then $\|\mu\|$ is its total variation whereas $\|\mu(S)\|$ is the absolute value of the number $\mu(s)$.

Composition will usually be denoted by $\circ$; thus if f is defined on the range of μ, then the function whose value on S is $f(\mu(S))$ will be denoted $f\circ\mu$; similarly if $\vec{\mu}$ is a vector-function and f a function of a vectorial variable on the range of $\vec{\mu}$. In the case of composition of linear operatiors, the symbol $\circ$ will be omitted.

The origin of a linear space will be denoted by 0 (sometimes for n-dimensional spaces it will be denoted by $\vec{0}$).

The symbol $\subset$ will be used for inclusion. Set theoretical subtraction will be denoted by $\setminus$, whereas $-$ will be reserved for algebraic subtraction. ∇ stands for the symmetric set-subtraction. $f|A$ will mean "f restricted to A". A^c means the complement of A in an appropriate space; if it is not clear which space we mean it will be pointed out.

A measure is an additive real valued set function defined on a field, which vanishes on $\emptyset$. It will be pointed out whether we mean a finitely additive or a σ-additive measure. A probability measure is a non-atomic σ-additive measure whose value on the entire space is 1. $|\mu|(S)$ means the total variation of μ on S. When μ is additive it is known that $|\mu|$ is additive too (e.g. [Dun-S] III-1-6, p. 98).

$([0,1], \mathcal{B})$ is the measurable space consisting of the closed unit interval and the σ-field of Borel.

Finally, if $x, y \in E^n$ then $\vec{x} \leqq \vec{y}$ iff $x_i \leqq y_i$ for each $1 \leq i \leq n$.

2. Basic Definitions and Conventions

This chapter is going to summarize definitions, conventions and results of [A-S] which we shall need.

Let $(I, \mathcal{C})$ be the measurable space consisting of the unit interval and the Borel subsets.* A set function is a real valued function v on $\mathcal{C}$ such that $v(\emptyset) = 0$. By a carrier of a set-function v we mean a set I' such that $v(S) = v(S \cap I')$ for each $S \in \mathcal{C}$. A set is null (or v-null) if it is the complement of a carrier. A set function is non-atomic if $\{s\}$ is null for each $s \in I$. A set function is monotonic if $S \subset T$ implies $v(S) \leq v(T)$. The difference between two monotonic set functions is said to be of bounded variation. The set of all set functions of bounded variation forms a linear space, which will be called BV. The linear subspace of BV consisting of all bounded finitely additive set functions will be denoted FA. Note that $\mu \in FA$ is monotonic iff $\mu(S) \geq 0$ for all S in $\mathcal{C}$. The subspace of all non-atomic σ-additive totally finite signed measures will be denoted by NA. The subspace of all σ-additive totally finite signed measures on $(I, \mathcal{C})$ will be denoted by M.

Let Q be a subspace of BV. The set of all monotonic set functions in Q is denoted Q^+. A mapping of Q into BV is called

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*This is assumed for simplicity only. All the results would remain true if $(I, \mathcal{C})$ is any countably generated and separated Borel space.

positive if it maps Q^+ into BV^+. If Q has no monotonic elements except 0 then every linear mapping is positive.

Let $\mathcal{G}$ denote the group of automorphisms of $(I, \mathcal{C})$, i.e., the one-one functions from I onto I which are measurable in both directions. Each θ in $\mathcal{G}$ induces a linear mapping θ_* of BV onto itself, given by

$$(\theta_* v)(S) = v(\theta S) \quad . \tag{2.1}$$

A subspace of BV is called symmetric if $\theta_* Q = Q$ for each θ in $\mathcal{G}$.

The norm we shall use in BV is the variation norm, defined by

$$\|v\| = \inf\{u(I) + w(I) \mid u-w = v, \text{ where } u \text{ and } w \text{ are monotonic}\} \quad .$$

Unless otherwise stated the norm in BV will always be the variation norm; it is easily seen that it is indeed a norm.

A chain is a non-decreasing sequence of sets of the form

$$\emptyset = S_0 \subset S_1 \subset \cdots \subset S_n = I \quad .$$

A link of this chain is a pair of successive elements. A subchain is a set of links. A chain will be identified with the subchain consisting of all links. If v is a set function and Λ is a subchain of a chain Ω, then the variation of v over Λ is defined by

$$\|v\|_\Lambda = \sum |v(S_i) - v(S_{i-1})| \quad ,$$

where the sum ranges over $\{i | \{S_{i-1}, S_i\} \in \Lambda\}$. For a fixed Λ, $\| \|_\Lambda$ is a pseudonorm on BV, i.e., it enjoys all the properties of a norm except $\|v\| = 0 \implies v = 0$. In [A-S] it is proved* that $v \in BV$ iff $\|v\|_\Omega$ is bounded over all chains Ω, and if $v \in BV$ then

$$\|v\| = \sup_\Omega \|v\|_\Omega \quad .$$

Clearly convergence in the variation norm implies pointwise convergence.

Let $v \in BV$ and $S \in \mathcal{C}$. Let v^S denote the restriction of v to the measurable subsets of S. Denote

$$|v|(S) = \|v^S\| \quad .$$

This coincides with the usual notation for the total variation of a measure. It is easily seen that

$$|v|(S) = \sup \sum_{i=1}^{k} |v(S_i) - v(S_{i-1})|$$

where the sup is taken over all non-decreasing sequences $\emptyset = S_0 \subset S_1 \subset \cdots \subset S_k = S$.

It is clear that the variation norm coincides with the usual norm for bounded finitely additive measures (see [Dun-S, 15, p. 140]). In [A-S, Proposition 4.3], it is shown that BV is complete in the

*Proposition 4.1.

variation norm. It can be proved straight-forwardly that NA, M and FA are closed in BV under the variation norm.

3. Weak Continuity

In Section 5 of [A-S], absolute continuity of a set function with respect to another set function is defined, as follows: Let v and w be set functions; then v is absolutely continuous with respect to w (written $v \ll w$) if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every chain Ω and every subchain Λ of Ω

$$\|w\|_\Lambda \leq \delta \implies \|v\|_\Lambda \leq \varepsilon \quad .$$

Note that the relation is transitive, and that if v and w are measures, it coincides with the usual notion of absolute continuity.

A set function is said to be absolutely continuous if there is a measure $\mu \in NA^+$ such that $v \ll \mu$. The set of all absolutely continuous set functions in BV is denoted AC. AC is a closed linear subspace of BV ([A-S, Proposition 5.2]).

A "weaker" concept of continuity was introduced in [A-S, Proof of 44.27]. Now we are going to define an even weaker continuity concept. For simplicity we will define this continuity with respect to members of FA only.

If $v \in BV$ and $\mu \in FA$ then v is said to be weakly continuous with respect to μ, written $v \underset{w}{<} \mu$, if for any $S,T \in \mathcal{C}$,

$$|\mu|(S \triangledown T) = 0 \implies v(S) = v(T) \quad . \tag{3.1}$$

Note that $v \underset{w}{<} \mu$ and $\mu \underset{w}{<} \eta$ where $v \in BV$ and $\mu, \eta \in FA^+$ implies $v \underset{w}{<} \eta$.

Lemma 3.2. If $v \in BV$, $\mu \in FA$ then the following statements are equivalent:

1. $v \underset{w}{<} \mu$.
2. If $S, T \in \mathcal{C}$ and $S \subset T$, then: $|\mu|(S) = |\mu|(T) \Longrightarrow v(S) = v(T)$.
3. If $S \in \mathcal{C}$ is μ-null then S is v-null.

Proof. $1 \Longrightarrow 2$: This is immediate.

$2 \Longrightarrow 3$: If S is μ-null then for any T, $|\mu|(T) = |\mu|(T \setminus (T \cap S))$, and 2 now yields $v(T) = v(T \setminus S)$.

$3 \Longrightarrow 1$: If $|\mu|(S \nabla T) = 0$ then $S \nabla T$ is μ-null and therefore by 3 v-null,

$$v(S) = v(S \setminus (S \nabla T)) = v(S \cap T) = v(T \setminus (S \nabla T)) = v(T) \ . \qquad \text{Q.E.D.}$$

Remark: Let $\vec{\mu}$ be an n-dimensional σ-additive measure whose components μ_i are in M^+. Let f be a real valued function on the range of $\vec{\mu}$ in E^n, with $f(\vec{0}) = 0$, and $f \circ \vec{\mu} \in BV$. Then $v \underset{w}{<} \sum |\mu_i|$.

A set function is said to be weakly continuous if there is a measure $\mu \in NA^+$ such that $v \underset{w}{<} \mu$. The set of all weakly continuous set functions in BV is denoted WC.

Proposition 3.3: WC is a closed linear symmetric subspace of BV.

Proof: WC is easily seen to be linear. By definition $WC \subset BV$. The symmetry follows immediately from

$$v \underset{w}{\leq} \mu \Longrightarrow \Psi_* v \underset{w}{\leq} \Psi_* \mu \qquad \text{for each automorphism } \Psi .$$

To prove WC is closed, let $\|v_i - v\| \underset{i\to\infty}{\to} 0$ where $v_i \underset{w}{\leq} \mu_i$ and $\mu_i \in NA^+$. Without loss of generality assume $\mu_i(I) = 1$ and set

$$\mu = \sum_{i=1}^{\infty} (\tfrac{1}{2})^i \mu_i .$$

Note that $\mu \in NA^+$, and $\mu_i \underset{w}{\leq} \mu$ for all i, hence $v_i \underset{w}{\leq} \mu$ for all i.

Let $S \subset T$ such that $\mu(S) = \mu(T)$. Now for a given $\varepsilon > 0$ let v_{i_0} be such that $\|v_{i_0} - v\| \leq \frac{\varepsilon}{2}$. Since $v_{i_0} \underset{w}{\leq} \mu$ we get

$$v_{i_0}(S) = v_{i_0}(T) .$$

Now

$$|v(T) - v(S)| \leq |v(T) - v_{i_0}(T)| + |v_{i_0}(T) - v_{i_0}(S)| + |(v_{i_0}(S) - v(S)|$$

$$\leq 2\|v - v_{i_0}\| + 0 \leq \varepsilon$$

ε was chosen arbitrarily therefore $v(T) = v(S)$. Hence $v \underset{w}{\leq} \mu$ and $v \in WC$ is proved. Q.E.D.

4. Measurable Orders

Measurable orders were introduced in [A-S, Section 12], for the purpose of trying to establish a notion of a value (in the game theoretic sense) based on "random orders".

Intuitively, each order* has a "direction". To emphasize this, orders will be denoted $x \underset{\mathcal{R}}{<} y$ instead of the usual $x \,\mathcal{R}\, y$, the intuitive meaning being that x comes before y. The notations $x \underset{\mathcal{R}}{\leq} y$, $x \underset{\mathcal{R}}{>} y$ will be used similarly.

The initial segment is a set of the form $I(s, \mathcal{R}) = \{x \mid x \underset{\mathcal{R}}{<} s\}$ where $s \in I$. A final segment is a set of the form $F(s, \mathcal{R}) = \{x \mid x \underset{\mathcal{R}}{>} s\}$ where $s \in I$. An initial set is a set J which fulfills the condition $s \in J$, $s' \underset{\mathcal{R}}{<} s \Longrightarrow s' \in J$. An $\mathcal{R}$-interval is a set of the form $[s,t)_{\mathcal{R}} = \{x \mid s \underset{\mathcal{R}}{\leq} x \underset{\mathcal{R}}{<} t\}$ where $s,t \in I$. The entire space and the empty set will also be considered as initial sets, and as such will be denoted $I(\infty, \mathcal{R})$, $I(-\infty, \mathcal{R})$ respectively; it will be understood $-\infty \underset{\mathcal{R}}{<} s \underset{\mathcal{R}}{<} \infty$ for each $s \in I$ and we will denote $\{-\infty\} \cup I \cup \{\infty\}$ by $\bar{I}$. (Formally we extend $\mathcal{R}$ to $\bar{I}$. This however is a notational device; we are not adding anything to the underlying space, and all set functions and measures continue to be defined on subsets of I only.)

Denote by $F(\mathcal{R})$ the σ-field generated by all the initial segments. A measurable order is an order such that $F(\mathcal{R}) = \mathcal{C}$.

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*An order on I is a relation on I that is transitive, irreflexive and complete.

A subset Q of I will be called $\mathcal{A}$-dense if for all $s,t \in I$ such that $s \underset{\mathcal{A}}{<} t$ there is a member $q \in Q$ such that $s \underset{\mathcal{A}}{<} q \underset{\mathcal{A}}{<} t$. By [A-S, Lemma 12.5], there exists a denumerable $\mathcal{A}$-dense set for any measurable order $\mathcal{A}$.

If Ψ is an automorphism of $(I, \mathcal{C})$, denoted by $\Psi\mathcal{A}$ the order defined by $\Psi x \underset{\Psi\mathcal{A}}{<} \Psi y$ iff $x \underset{\mathcal{A}}{<} y$. Obviously, $\Psi\mathcal{A}$ is measurable iff $\mathcal{A}$ is.

<u>Lemma 4.1</u>: Let $\mathcal{A}$ be a measurable order. Let $A \in \mathcal{C}$. Define an order $\mathcal{A}^*$ by

$$x \underset{\mathcal{A}^*}{<} y \iff \begin{cases} x \in A,\ y \in A & \text{and} \quad x \underset{\mathcal{A}}{<} y\ , \quad \text{or} \\ x \notin A,\ y \notin A & \text{and} \quad x \underset{\mathcal{A}}{<} y\ , \quad \text{or} \\ x \notin A,\ y \in A; & \end{cases}$$

(this means A is "thrown" beyond $I \setminus A$ and the order $\mathcal{A}$ is preserved on A and $I \setminus A$). Then $\mathcal{A}^*$ is measurable.

<u>Proof</u>: The direction $F(\mathcal{A}^*) \subset \mathcal{C}$ is trivial. To prove the opposite we shall first show that there is a denumerable $\mathcal{A}^*$-dense set. Let Q be a $\mathcal{A}$-dense denumerable set ([A-s, Lemma 12.5] assures its existence). Denote

$$B_q = \begin{cases} I \setminus A & q \in A \\ A & q \in I \setminus A \end{cases} .$$

We have to substitute for q in B_q. If there is a minimal element in $F(q, \mathcal{R}) \cap B_q$ or a maximal element in $I(q, \mathcal{R}) \cap B_q$ then we might substitute it for q in B_q. Otherwise we might find a sequence which approaches in B_q the place where q has been. Denote by I_q the element or the sequence which substitute for q in B_q, then $Q' = \bigcup_Q I_q \cup Q$ is a denumerable $\mathcal{R}^*$ dense set.

Let J be a $\mathcal{R}^*$-initial set, we shall show $J \in F(\mathcal{R}^*)$. Denote

$$\bar{Q}' = Q' \cup \{-\infty\} \cup \{\infty\} \qquad \bar{J} = J \cup \{-\infty\}$$

$$J_1 = \bigcap_{q \in \bar{Q}' \setminus J} I(q, \mathcal{R}^*)$$

$J_1 \supset J$ and because Q' is $\mathcal{R}^*$ dense it follows $J_1 \setminus J$ contains at most two points. Since the intersection defining J_1 is denumerable it follows $J_1 \in F(\mathcal{R}^*)$ and this assures $J \in F(\mathcal{R}^*)$.

Now clearly $I(x, \mathcal{R}) \in F(\mathcal{R}^*)$ for each x in I. This is sufficient to show $\mathcal{C} \subset F(\mathcal{R}^*)$, since $\mathcal{C} = F(\mathcal{R})$.

<u>Corollary 4.2</u>: Let $A_1 \cdots A_n$ be distinct measurable sets whose union is I. Then there is a measurable order $\mathcal{R}$ such that

(4.3) $$x \in A_i, \; y \in A_j, \text{ and } i < j \implies x \underset{\mathcal{R}}{<} y \quad .$$

Proof: Start with the usual order on $[0,1]$, which is clearly measurable. Define an order which "throws" A_1 beyond $[0,1]\setminus A_1$ and preserves $\mathcal{R}$ on A_1 and $I\setminus A_1$. By repeating this for $A_2, \ldots, A_n$ subsequentially we shall get an order which satisfies (4.3). Q.E.D.

Corollary 4.4: Let $A_1 \cdots A_n$ be distinct measurable sets whose union is I. Then there exits a measurable order $\mathcal{R}$ such that (4.3) is satisfied and there is an $\mathcal{R}$-minimal element in each A_i.

Proof: Choose $x_i \in A_i$ for each i. Denote

$$B_{2i-1} = \{x_i\} \qquad B_{2i} = A_i\setminus\{x_i\}$$

Using Corollary 4.2, for B_j, $1 \le j \le 2n$ proves our claim. Q.E.D.

We shall use the notation $A \underset{\mathcal{R}}{<} B$ when $x \underset{\mathcal{R}}{<} y$ for each $x \in A$ and $y \in B$.

5. Orderable Set Functions

A set function v is called orderable [A-S, Section 12] if for each measurable order $\mathcal{R}$ there is a σ-additive measure $\varphi^{\mathcal{R}}v$ such that for all initial segments $I(s, \mathcal{R})$, we have

$$(5.1) \qquad (\varphi^{\mathcal{R}}v)\,(I(s, \mathcal{R})) = v(I(s, \mathcal{R})) \quad .$$

Since (5.1) determines $(\varphi^{\mathcal{R}}v)$ on all the initial segments, and by the measurability of $\mathcal{R}$ the initial segments generate $\mathcal{C}$, it follows that there can be at most one measure $\varphi^{\mathcal{R}}v$ satisfying (5.1). Thus for orderable set functions there is exactly one measure $\varphi^{\mathcal{R}}v$ satisfying (5.1). The set of all orderable set functions will be denoted ORD.

Proposition 5.2. 1. ORD is a closed linear symmetric subspace of BV.

2. For all meausrable orders $\mathcal{R}$, $\varphi^{\mathcal{R}}$ is a bounded linear operator on ORD. Moreover $\|\varphi^{\mathcal{R}}\| = 1$.

3. $\varphi^{\mathcal{R}}\psi_* v = \psi_* \varphi^{\psi\mathcal{R}} v$.

Proof. Obviously ORD is linear and for all measurable orders $\mathcal{R}$, $\varphi^{\mathcal{R}}$ is linear on ORD. Let $\mathcal{R}$ be a measurable order and let ψ be an automorphism of $(I, \mathcal{C})$. Then $\psi_* \varphi^{\psi\mathcal{R}}v$ is a σ-additive measure. Furthermore

$$\psi(I(s, \mathcal{R})) = I(\psi s, \psi\mathcal{R}) \ ,$$

and hence

$$\psi_* \varphi^{\psi\mathcal{R}} v(I(s, \mathcal{R})) = \varphi^{\psi\mathcal{R}} v(\psi(I(s, \mathcal{R})))$$

$$= \varphi^{\psi\mathcal{R}} v(I(\psi s, \psi\mathcal{R})) = v(I(\psi s, \psi\mathcal{R}))$$

$$= v(\psi(I(s, \mathcal{R}))) = (\psi_* v)\ (I(s, \mathcal{R})) \quad ,$$

since $\psi_* \varphi^{\psi\mathcal{R}} v$ is a σ-additive measure satisfying (5.1) for $\psi_* v$ and $\mathcal{R}$ it follows $\varphi^{\mathcal{R}} \psi_* v$ exists and equals $\psi_* \varphi^{\psi\mathcal{R}} v$. Hence $\psi_* v \in$ ORD.

Next, let $\mathcal{R}$ be a measurable order. In [A-S, Proposition 12.8] it has been proved that $\|\varphi^{\mathcal{R}} v\| \leq \|v\|$, which shows that $\|\varphi^{\mathcal{R}}\| \leq 1$. To see $\|\varphi^{\mathcal{R}}\| \geq 1$ let μ be any probability measure on $\mathcal{C} = F(\mathcal{R})$, then clearly $\varphi^{\mathcal{R}} \mu = \mu \neq 0$. This shows $\|\varphi^{\mathcal{R}}\| = 1$.

Finally we will prove that ORD is closed. Let $\mathcal{R}$ be a measurable order and let $v_n \to v$ as $n \to \infty$, where $v_n \in$ ORD for each n. Then

$$\|\varphi^{\mathcal{R}} v_n - \varphi^{\mathcal{R}} v_m\| = \|\varphi^{\mathcal{R}}(v_n - v_m)\| \leq \|(v_n - v_m)\|_{n,m\to\infty} \to 0 \ .$$

Thus, we see that $\varphi^{\mathcal{R}} v_n$ is a Cauchy sequence. As BV is complete in the variational norm [A-S, Section 4], it follows that $\varphi^{\mathcal{R}} v_n$ converges; denote its limit by η. The set of σ-additive measures is closed in BV, and therefore η is a σ-additive measure. Moreover, convergence in the variational norm clearly implies pointwise convergence, and therefore

$$\eta(I(s, \mathcal{R})) = \lim_{n \to \infty} \varphi^{\mathcal{R}} v_n(I(s, \mathcal{R})) = \lim_{n \to \infty} v_n(I(s, \mathcal{R})) = v(I(s, \mathcal{R}))$$

Hence η is a σ-additive measure which fulfills (5.1). Hence $v \in$ ORD has been proved. Q.E.D.

Corollary 5.3. Let $\mathcal{R}$ be a measurable order then $\varphi^{\mathcal{R}}$ is positive on ORD.

Proof. If Q is a linear subspace of BV and $\mathcal{R}$ a linear operator from Q into $M \subset$ BV obeying the normalization condition $(\varphi v)(I) = v(I)$ and $\|\varphi\| \leq 1$ then φ is positive [A-S, Proposition 4.6]. This fact and Proposition 5.2 complete the proof. Q.E.D.

Proposition 5.4. Let $v \in$ BV such that there is a measure μ for which $v << \mu$, then $v \in$ ORD.

Proof. See [A-S, Proposition 12.8], and note that the non-atomicity of μ was actually not used. Q.E.D.

We are going now to investigate properties which are inherited by $\varphi^{\mathcal{R}} v$ from v. First we will look at the non-atomic set functions in ORD (Propositions 5.6, 5.9, and 5.10) and afterwards at the general case (Propositions 5.11, 5.14, and 5.15).

Lemma 5.5. Let $v \in$ ORD and let $S_1 \subset T_1 \subset S_2 \subset T_2 \subset \cdots \subset S_n \subset T_n$, where $S_i, T_i \in \mathcal{C}$ for $1 \leq i \leq n$. Then there is a measurable order $\mathcal{R}$ such that

$$(\varphi^{\mathcal{R}} v)\{\bigcup_{i=1}^{n} (T_i \setminus S_i)\} = \sum_{i=1}^{n} v(T_i) - v(S_i) \quad .$$

Proof. Let us define

$$A_{2k} = T_k \setminus S_k \qquad A_{2k+1} = S_{k+1} \setminus T_k \qquad \text{for } k \geq 1$$

$$A_{2n+1} = I \setminus T_n \qquad A_0 = \phi \qquad A_1 = S_1$$

Corollary 4.4 assures the existence of a measurable order $\mathcal{R}$ such that

$$A_1 \underset{\mathcal{R}}{<} A_2 \underset{\mathcal{R}}{<} A_3 \underset{\mathcal{R}}{<} \cdots \underset{\mathcal{R}}{<} A_{2n+1} \quad ,$$

and every set has an $\mathcal{R}$-first element which will be called x_i

$$(\varphi^{\mathcal{R}} v)\{\bigcup_{i=1}^{n} (T_i \setminus S_i)\} = \sum_{i=1}^{n} (\varphi^{\mathcal{R}} v)([x_{2i}, x_{2i+1})_{\mathcal{R}})$$

$$= \sum_{i=1}^{n} v(I(x_{2i+1}, \mathcal{R})) - v(I(x_{2i}, \mathcal{R})) = \sum_{i=1}^{n} (v(T_i) - v(S_i)) \quad .$$

Q.E.D.

Proposition 5.6. Let $v \in$ ORD. Then, v is non-atomic (if and only if) for any measurable order $\mathcal{R}$, $\varphi^{\mathcal{R}} v$ is non-atomic.

<u>Remark</u>. The conclusion need not hold if we do not assume $v \in \mathrm{ORD}$ -- even if we do assume that for the $\mathcal{R}$ in question, there is a σ-additive measure $\varphi^{\mathcal{R}}v$ satisfying (5.1)! For example let $v = f \circ \lambda$, where λ is Lebesgue measure and

$$f(x) = \begin{cases} 0 & x \leq \frac{1}{2} \\ 1 & x > \frac{1}{2} \end{cases}$$

and $\mathcal{R}$ be the usual order (which is obviously measurable); it is clear $\varphi^{\mathcal{R}}v$ exists and equals to the measure concentrated at $\frac{1}{2}$ which is not non-atomic. It might easily be shown that $v \notin \mathrm{ORD}$ for denote by $\mathcal{R}'$ the order which throws $\frac{1}{2}$ beyond $[0,1]$ and coincides with the usual order on $[0,1] \setminus \{\frac{1}{2}\}$. $\mathcal{R}'$ is measurable (Lemma 4.1). If $\varphi^{\mathcal{R}'}v$ existed then for $n \geq 3$ $(\varphi^{\mathcal{R}'}v)\ ([\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n})_{\mathcal{R}'}) = 1$ in spite of the fact $[\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n})_{\mathcal{R}'}$ is a decreasing sequence with a void intersection.

<u>Proof:</u> If v is not non-atomic then there exists a $s \in I$ and a set $T \in \mathcal{C}$ such that $v(T \setminus \{s\}) \neq v(T)$. Looking at the chain $\phi \subset T \setminus \{s\} \subset T \subset I$ and using Lemma 5.5 we know that there exists a measurable order $\mathcal{R}$ such that $(\varphi^{\mathcal{R}}v)(T \setminus (T \setminus \{s\})) = v(T) - v(T \setminus \{s\}) \neq 0$, i.e., $\varphi^{\mathcal{R}}v(\{s\}) \neq 0$. This contradicts the non-atomicity of $\varphi^{\mathcal{R}}v$.

For the opposite direction, let v in ORD be non-atomic, and $\mathcal{R}$ be a measurable order. $\varphi^{\mathcal{R}}v$ is σ-additive, therefore it is sufficient to prove $\varphi^{\mathcal{R}}v(\{s\}) = 0$ for all $s \in I$. Let $s \in I$ be fixed. Henceforth greater or smaller will be w.r.t. the order $\mathcal{R}$. Q will denote a denumerable $\mathcal{R}$-dense subset whose existence follows from [A-S, Proposition 12.4], $\bar{Q} = \{-\infty\} \cup Q \cup \{\infty\}$.

First Case. If there is a $\mathcal{R}$-minimal element in $F(s, \mathcal{R})$ (remember $F(s, \mathcal{R}) = \{x \mid x \underset{\mathcal{R}}{\geq} s\}$), let it be a. Then $\{s\} = [s,a)_{\mathcal{R}}$ and our conclusion becomes trivial. (This case holds when s is the greatest element in I, then $a = \infty$).

Second Case. In this case we assume that there exists no $\mathcal{R}$-maximal element in $I(s, \mathcal{R})$ and that there does not exist a $\mathcal{R}$-minimal element in $F(s, \mathcal{R})$. W.l.o.g we may assume that $s \notin Q$ (if $s \in Q$ change Q appropriately). Let

$$J = I(s, \mathcal{R}) \cup \{s\} \qquad \bar{J} = J \cup \{-\infty\}$$

$$J_1 = \bigcap_{q \in \bar{Q} \setminus \bar{J}} I(q, \mathcal{R}) \qquad J_0 = \bigcup_{q \in \bar{Q} \cap \bar{J}} J(q, \mathcal{R})$$

Then $J_1 \supset J \supset J_0$. The facts that Q is $\mathcal{R}$-dense and that there is no $\mathcal{R}$-maximal element in $I(s, \mathcal{R})$ imply that $J = J_0 \cup \{s\}$. Since there is no $\mathcal{R}$-minimal element in $F(s, \mathcal{R})$ and $s \notin Q$ it follows $J_1 = J$, hence $J_1/J_0 = \{s\}$.

Since the intersection and union defining J_1 and J_0 respectively are denumerable it follows J_0, J_1 are measurable. Furthermore since $I(q, \mathcal{R})$ are linearly ordered under inclusion each finite intersection equals to one of the $I(q, \mathcal{R})$; hence

$$J_1 = \bigcap_{i=1}^{\infty} I(q_i^1, \mathcal{R})$$

where $\{q_i^1\}$ is a $\mathcal{R}$-decreasing sequence of points in $\bar{Q} - \bar{J}$; i.e., $I(q_i^1, \mathcal{R})$ is a decreasing set sequence. Similarly

$$J_0 = \bigcup_{i=1}^{\infty} I(q_i^o, \mathcal{R})$$

where $\{q_i^o\}$ is a $\mathcal{R}$ increasing sequence of points in $\bar{Q} \cap \bar{J}$; i.e., $I(q_i^o, \mathcal{R})$ is a increasing set sequence. Obviously

$$J_1 \setminus J_0 = \bigcap_{i=1}^{\infty} I(q_i^1, \mathcal{R}) \setminus \bigcup_{i=1}^{\infty} I(q_i^o, \mathcal{R}) = \bigcap_{i=1}^{\infty} \{I(q_i^1, \mathcal{R}) \setminus I(q_i^o, \mathcal{R})\}$$

Note that $I(q_i^1, \mathcal{R}) \setminus I(q_i^o, \mathcal{R})$ is a decreasing sequence, and $\varphi^{\mathcal{R}}v$ is a totally finite σ-additive measure; this yields

$$(5.7)\quad (\varphi^{\mathcal{R}}v)\{s\} = (\varphi^{\mathcal{R}}v)(J_1 \setminus J_0) = \lim_{i \to \infty} (\varphi^{\mathcal{R}}v)\{I(q_i^1, \mathcal{R}) \setminus I(q_i^o, \mathcal{R})\}$$

$$= \lim_{i \to \infty} (v(I(q_i^1, \mathcal{R})) - v(I(q_i^o, \mathcal{R}))) \quad .$$

Define an order $\mathcal{R}^*$ which "throws" s beyond all other elements and preserves $\mathcal{R}$ on $I - \{s\}$ (Lemma 4.1 assures its measurability). Denote

$$J_1^* = \bigcap_{i=1}^{\infty} I(q_i^1, \mathcal{R}^*) \qquad J_0^* = \bigcup_{i=1}^{\infty} I(q_i^o, \mathcal{R}^*) \quad .$$

Since $s \notin Q$, we get that

$$\{s\} \cup I(q_i^1, \mathcal{R}^*) = I(q_i^1, \mathcal{R}), \qquad I(q_i^o, \mathcal{R}^*) = I(q_i^o, \mathcal{R}) \quad ,$$

this yields that

$$\{s\} \cup J_1^* = J_1 \qquad J_0^* = J_0 \quad .$$

Now, since $J_1 = J = J_0 \cup \{s\}$ it follows $J_1^* = J_0^*$, hence

$$\phi = J_1^* \setminus J_0^* = \bigcap_{i=1}^{\infty} I(q_i^1, \mathcal{R}^*) \setminus \bigcup_{i=1}^{\infty} I(q_i^o, \mathcal{R}^*)$$

$$= \bigcap_{i=1}^{\infty} I(q_i^1, \mathcal{R}^*) \setminus I(q_i^o, \mathcal{R}^*)$$

Since $I(q_i^1, \mathcal{R}^*) \setminus I(q_i^o, \mathcal{R}^*)$ is a decreasing sequence and $\varphi^{\mathcal{R}^*} V$ is a totally finite σ-additive measure we get

$$0 = (\varphi^{\mathcal{R}^*} v)(J_1^* \setminus J_0^*) = \lim_{i \to \infty} (\varphi^{\mathcal{R}^*} v)\{I(q_i^1, \mathcal{R}^*) \setminus I(q_i^o, \mathcal{R}^*)\}$$

$$= \lim_{i \to \infty} (v(I(q_i^1, \mathcal{R}^*)) - v(I(q_i^o, \mathcal{R}^*)))$$

$$= \lim_{i \to \infty} (v(I(q_i^1, \mathcal{R}) \cup \{s\}) - v(I(q_i^o, \mathcal{R})))$$

$$= \lim_{i \to \infty} (v(I(q_i^1, \mathcal{R})) - v(I(q_i^o, \mathcal{R})))$$

$$= (\varphi^{\mathcal{R}} v)\{s\}$$

(we used (5.7) and the non-atomicity of v in the last two equalities).

<u>Third case</u>. In this case we assume there is a $\mathcal{R}$-maximal element - b_1 in $I(s, \mathcal{R})$, but there is no $\mathcal{R}$-minimal element in $F(s, \mathcal{R})$. W.l.o.g. assume again that $s \notin 0$. Let

$$J = I(s, \mathcal{R}) \cup \{s\} = I(b_1, \mathcal{R}) \cup \{b_1, s\}$$

$$\bar{J} = J \cup \{-\infty\}$$

$$J_1 = \bigcap_{q \in \overline{Q} \setminus \bar{J}} I(q, \mathcal{R}) \quad .$$

The same arguments used in the second case lead us to the conclusion that J_1 is measurable and $J_1 = \bigcap_{i=1}^{\infty} I(q_i^1, \mathcal{R})$, where q_i^1 is a $\mathcal{R}$-decreasing sequence. As there is no $\mathcal{R}$-minimal element in $F(s, \mathcal{R})$ and $s \notin Q$ it follows $J_1 = J$. Note that $I(q_i^1, \mathcal{R})$ is a decreasing sequence of sets and $\varphi^{\mathcal{R}} v$ is a totally finite σ-additive measure. Using the first case with respect to b_1 we get

$$(5.8) \quad (\varphi^{\mathcal{R}} v)\{s\} = (\varphi^{\mathcal{R}} v)\{b_1, s\} = (\varphi^{\mathcal{R}} v)(J_1) - (\varphi^{\mathcal{R}} v)(I(b_1, \mathcal{R}))$$

$$= \lim_{i \to \infty} v(I(q_i^1, \mathcal{R})) - v(I(b_1, \mathcal{R})) \quad .$$

Define a measurable order $\mathcal{R}_1$ which "throws" s beyond all other elements. (Lemma 4.1 assures its measurability.) Let

$$J_1^{(1)} = \bigcap_{i=1}^{\infty} I(q_i^1, \mathcal{R}_1) \quad .$$

Since $I(q_i^1, \mathcal{R}_1) \cup \{s\} = I(q_i^1, \mathcal{R})$ for all $i \geq 1$ it follows

$$J_1^{(1)} \cup \{s\} = J_1 = J \quad .$$

Now, by using (5.8) and the nonatomicity of v we get that

$$(\varphi^{\mathcal{R}} v)\{s\} = \lim_{i \to \infty} v(I(q_i^1, \mathcal{R})) - v(I(b_1, \mathcal{R}))$$

$$= \lim_{i \to \infty} v(I(q_i^1, \mathcal{R}_1)) - v(I(b_1, \mathcal{R}_1))$$

$$= (\varphi^{\mathcal{R}_1} v) \left(\bigcap_{i=1}^{\infty} I(q_i^1, \mathcal{R}_1)\right) - (\varphi^{\mathcal{R}_1} v) (I(b_1, \mathcal{R}))$$

$$= (\varphi^{\mathcal{R}_1} v) (J_1^{(1)} \setminus I(b_1, \mathcal{R}_1)) = (\varphi^{\mathcal{R}_1} v) \{b_1\} \quad .$$

Clearly there is no $\mathcal{R}_1$-minimal element in $F(b_1, \mathcal{R}_1)$. If there is no $\mathcal{R}_1$-maximal element in $I(b_1, \mathcal{R}_1)$ (or equivalently in $I(b_1, \mathcal{R})$), then the proof of the second case yields that

$$0 = (\varphi^{\mathcal{R}_1} v)\{b_1\} = (\varphi^{\mathcal{R}} v)\{s\} \quad .$$

If there is such an element, denote it by b_2. Define an order $\mathcal{R}_2$ which throws b_1 beyond all other elements, we shall get analoguously

$$(\varphi^{\mathcal{R}_2} v)(\{b_2\}) = (\varphi^{\mathcal{R}_1} v) \ (\{b_1\}) = (\varphi^{\mathcal{R}} v) \ (\{s\}) \quad .$$

Going on in this way we shall build a sequence of measurable orders $\mathcal{R}_n$ and a sequence of elements in $I - b_n$, such that $\mathcal{R}_{n+1}$ is the measurable order which is obtained from $\mathcal{R}_n$ by "throwing" b_n beyond all other elements and b_{n+1} is the $\mathcal{R}_n$-maximal element in $I(b_n, \mathcal{R}_n) = I(b_n, \mathcal{R})$ (provided it exists). We shall get

$$(\varphi^{\mathcal{R}_{n+1}})(\{b_{n+1}\}) = (\varphi^{\mathcal{R}_n})(\{b_n\}) = (\varphi^{\mathcal{R}}v)(\{s\}) \quad .$$

If at any step we get there is no $\mathcal{R}_n$-maximal element in $I(b_n, \mathcal{R}_n) = I(b_n, \mathcal{R})$, the proof of the second case yields $0 = (\varphi^{\mathcal{R}_n}v)(\{b_n\}) = (\varphi^{\mathcal{R}}v)(\{s\})$. If for some n, $b_n = -\infty$ we easily get $0 = \varphi^{\mathcal{R}_n}v(\{b_n\}) = \varphi^{\mathcal{R}}v(\{s\})$. Otherwise the procedure goes on for all $n \geq 1$. In this case define an order $\mathcal{R}^*$

$$x \underset{\mathcal{R}^*}{<} y \iff \begin{cases} x,y \in \{b_n\}_{n\geq 1} & y \underset{\mathcal{R}}{<} x \\ \text{otherwise} & x \underset{\mathcal{R}}{<} y \end{cases}$$

this means we reverse the order $\mathcal{R}$ on b_n and preserve it anywhere else. Note that $\mathcal{R}^*$ is measurable and that $\{s\} = J_1 \setminus I(s, \mathcal{R}) = J_1 \setminus I(s, \mathcal{R}^*)$. Clearly

$$(\varphi^{\mathcal{R}^*}v)(J_1) = \lim_{i \to \infty} (\varphi^{\mathcal{R}^*}v)(I(q_i^1, \mathcal{R}^*)) = \lim_{i \to \infty} (\varphi^{\mathcal{R}}v)(I(q_i^1, \mathcal{R})) \ ,$$

hence

$$\begin{aligned} (\varphi^{\mathcal{R}^*}v)(\{s\}) &= (\varphi^{\mathcal{R}^*}v)(J_1) - (\varphi^{\mathcal{R}^*}v)(I(s, \mathcal{R}^*)) \\ &= \lim_{i \to \infty} (\varphi^{\mathcal{R}}v)(I(q_i^1, \mathcal{R})) - v(I(s, \mathcal{R}^*)) \\ &= (\varphi^{\mathcal{R}}v)(J_1) - v(I(s, \mathcal{R})) \\ &= (\varphi^{\mathcal{R}}v)(J_1) - (\varphi^{\mathcal{R}}v)(I(s, \mathcal{R})) \\ &= (\varphi^{\mathcal{R}}v)(\{s\}) \quad . \end{aligned}$$

Clearly there is no $\mathcal{R}^*$-minimal element in $F(s, \mathcal{R}^*)$, and no $\mathcal{R}^*$-maximal element in $I(s, \mathcal{R}^*)$ therefore the second case assures

$$0 = (\varphi^{\mathcal{R}^*} v)(\{s\}) = (\varphi^{\mathcal{R}} v)(\{s\}) \quad . \qquad \text{Q.E.D.}$$

<u>Proposition 5.9</u>. Let $v \in \mathrm{ORD}$, $\mu \in \mathrm{NA}^+$, Then $v \underset{w}{<} \mu \iff \varphi^{\mathcal{R}} v \underset{w}{<} \mu$ for each measurable order $\mathcal{R}$.

<u>Remark.</u> If $v \notin \mathrm{ORD}$ we cannot assure $v \underset{w}{<} \mu \Longrightarrow \varphi^{\mathcal{R}} v \underset{w}{<} \mu$ even if $\varphi^{\mathcal{R}} v$ happens to be defined (by (5.1)). Indeed, let λ be the Lebesgue measure and let $v = f \circ \lambda$, where

$$f(x) = \begin{cases} 0 & x \le \frac{1}{2} \\ 1 & x > \frac{1}{2} \end{cases} \quad .$$

If $\mathcal{R}$ is the usual order on $[0,1]$, then $\varphi^{\mathcal{R}} v$ is the measure concentrated at $\frac{1}{2}$; now $v \underset{w}{<} \lambda$, but $\varphi^{\mathcal{R}} v \underset{w}{<} \lambda$ does not hold.

<u>Proof.</u> $\Leftarrow$: If $v \underset{w}{<} \mu$ does not hold then there exist $S, T \in \mathcal{C}$, $S \subset T$ where $\mu(T) = \mu(S)$ and $v(T) \neq v(S)$. Looking at the chain $\phi \subset S \subset T \subset I$ and using Lemma 5.5, we know that there exists a measurable order $\mathcal{R}$ such that $(\varphi^{\mathcal{R}} v)(T \backslash S) = v(T) - v(S) \neq 0$. This contradicts the assumption that $\varphi^{\mathcal{R}} v \underset{w}{<} \mu$.

$\Rightarrow$: $\varphi^{\mathcal{R}} v \underset{w}{<} \mu$ means usual continuity of a measure with respect to another measure (see [Dun-S, p. 131]).

Let $H(\mathcal{R})$ be the field (not σ-field) generated by the initial segments $I(s; \mathcal{R})$. If two finitely additive measures μ_1, λ_1 are defined on a field Σ_1 and there exist countably additive extensions μ_2, λ_2 of μ_1, λ_1 respectively to the σ-field generated by Σ_1, then λ_1 is μ_1-continuous iff λ_2 is μ_2-continuous [Dun-S, IV 9-13,

p. 315]. $\varphi^{\mathfrak{A}}v$ and μ are extensions to $\mathcal{C}$ of $\varphi^{\mathfrak{A}}v|H(\mathfrak{A})$ and $\mu|H(\mathfrak{A})$ respectively. If $\varphi^{\mathfrak{A}}v|H(\mathfrak{A})$ is not $\mu|H(\mathfrak{A})$-continuous then there is an ϵ such that for each $n \geq 1$ there exists a finite sequence

$$s_1^{(n)} \underset{\mathfrak{A}}{<} t_1^{(n)} \underset{\mathfrak{A}}{<} s_2^{(n)} \cdots s_{k_n}^{(n)} \underset{\mathfrak{A}}{<} t_{k_n}^{(n)}$$

such that

$$\mu\{\bigcup_{i=1}^{k_n} [s_i^{(n)}, t_i^{(n)})_{\mathfrak{A}}\} < \frac{1}{2^n} \quad \text{and} \quad |(\varphi^{\mathfrak{A}}v)\{\bigcup_{i=1}^{k_n} [s_i^{(n)}, t_i^{(n)})\}| \geq \epsilon \quad .$$

Denoting

$$W_i^{(n)} = [s_i^{(n)}, t_i^{(n)})_{\mathfrak{A}} \qquad W_n = \bigcup_{i=1}^{k_n} W_i^{(n)}$$

$$A_n = \bigcup_{k=n}^{\infty} W_k \qquad A = \bigcap_{n=1}^{\infty} A_n \quad ,$$

we get

$$\mu(A_n) \leq \sum_{k=n}^{\infty} \frac{1}{2^k} = \frac{1}{2^{n-1}}$$

$$|\varphi^{\mathfrak{A}}v|(A_n) \geq |\varphi^{\mathfrak{A}}v|(W_n) \geq |(\varphi^{\mathfrak{A}}v)(W_n)| \geq \epsilon \quad .$$

If it happens that A is empty or denumerable then $|\varphi^{\mathfrak{A}}v|(A) = 0$ (Proposition 5.6 assures $|\varphi^{\mathfrak{A}}v|$ is non-atomic, since $v \in WC$ and

therefore v is non-atomic). But $A = \bigcap_{n=1}^{\infty} A_n$, and A_n is a decreasing sequence, and $\varphi^{\mathfrak{A}} v$ is totally finite; therefore

$$0 = |\varphi^{\mathfrak{A}} v|(A) = \lim_{n \to \infty} |\varphi^{\mathfrak{A}} v|(A_n) \geq \varepsilon \quad ,$$

and this contraction completes the proof in this case.

If A is non-denumerable we still have $\mu(A) = \lim_{n \to \infty} \mu(A_n) = 0$, and therefore A is a v-null set since $v \underset{w}{<} \mu$.

Let us define an order $\mathfrak{A}^*$ that throws

$$A' = A \setminus \{s_i^{(n)}, t_i^{(n)} | 1 \leq i \leq k_n, n \geq 1\}$$

beyond $I \setminus A'$ and preserves $\mathfrak{A}$ on A' and $I \setminus A'$. From Lemma 4.1 it follows that $\mathfrak{A}^*$ is measurable. Define

$$W_i^{(n)*} = [s_i^{(n)}, t_i^{(n)})_{\mathfrak{A}^*} \qquad W_n^* = \bigcup_{i=1}^{k_n} W_i^{(n)*} \quad ,$$

$$A_n^* = \bigcup_{k=n}^{\infty} W_k^* \qquad A^* = \bigcap_{n=1}^{\infty} A_n^* \quad .$$

Clearly $I(a, \mathfrak{A}^*) = I(a, \mathfrak{A}) \setminus A'$ for each $a \in I \setminus A'$, in particular for $a = s_i^{(n)}, t_i^{(n)}$. Therefore

$$W_i^{(n)*} = W_i^{(n)} \setminus A' \qquad W_n^* = W_n \setminus A'$$

$$A_n^* = A_n \setminus A' \qquad A^* = A \setminus A' \quad .$$

Since A is v-null and $A' \subset A$, A' is v-null; therefore

$$(\varphi^{\mathfrak{A}^*}v)(W_i^{(n)*}) = (\varphi^{\mathfrak{A}^*}v)([s_i^{(n)}, t_i^{(n)})_{\mathfrak{A}^*})$$

$$= v(I(t_i^{(n)}, \mathfrak{A}^*) - v(I(s_i^{(n)}, \mathfrak{A}^*))$$

$$= v(I(t_i^{(n)}, \mathfrak{A})) - v(I(s_i^{(n)}, \mathfrak{A}))$$

$$= (\varphi^{\mathfrak{A}}v)([s_i^{(n)}, t_i^{(n)})_{\mathfrak{A}})$$

$$= (\varphi^{\mathfrak{A}}v)(W_i^{(n)}) \quad .$$

Since $\mu(A') = 0$, we have

$$|\mu(W_n^*)| = |\mu(W_n \setminus A')| = |\mu(W_n)| \leq \frac{1}{2^n}$$

$$|(\varphi^{\mathfrak{A}^*}v)(W_n^*)| = |\sum_{i=1}^{k_n} (\varphi^{\mathfrak{A}^*}v)(W_i^{(n)*})|$$

$$= |\sum_{i=1}^{k_n} (\varphi^{\mathfrak{A}}v)(W_i^{(n)})| = |(\varphi^{\mathfrak{A}}v)(W_n)| \geq \varepsilon \quad .$$

Since A^* is contained in $\{s_i^{(n)}, t_i^{(n)} | 1 \leq i \leq k_n, n \geq 1\}$, it is denumerable; therefore $|\varphi^{\mathfrak{A}^*}v|(A^*) = 0$, ($\varphi^{\mathfrak{A}^*}v$ is non-atomic by Proposition 5.6). Now A_n^* is decreasing, $A^* = \bigcap_{n=1}^{\infty} A_n^*$ and $|\varphi^{\mathfrak{A}}v|$ is totally finite; therefore

$$0 = |\varphi^{\mathcal{R}^*} v|(A^*) = \lim_{n \to \infty} |\varphi^{\mathcal{R}^*} v|(A_n^*) \geq \overline{\lim_{n \to \infty}} |\varphi^{\mathcal{R}^*} v|(W_n^*)$$

$$\geq \overline{\lim_{n \to \infty}} |(\varphi^{\mathcal{R}^*} v)(W_n^*)| \geq \epsilon \quad .$$

This contradiction completes the proof. Q.E.D.

<u>Proposition 5.10</u>: (Aumann) Let $v \in ORD \cap WC$. Then A is v-null $\Longleftrightarrow$ A is $\varphi^{\mathcal{R}}v$-null for each measurable order $\mathcal{R}$.

<u>Remark</u>: It is not known whether a non-atomic orderable set function is necessarily in WC. If this is so, then of course we may replace the hypothesis: "$v \in ORD \cap WC$" in the above Proposition by: "v is non-atomic and orderable".

If $v \notin ORD$, the conclusion need not follow even if $\varphi^{\mathcal{R}}v$ happens to be defined (by (5.1)). Take the example in the preceding remark; then $\varphi^{\mathcal{R}}v(\{\frac{1}{2}\}) = 1$ but $\frac{1}{2}$ is a v-null set.

<u>Proof</u> $\Longleftarrow$: If A is $\varphi^{\mathcal{R}}v$-null for each measurable order $\mathcal{R}$ then $|\varphi^{\mathcal{R}}v|(A) = 0$ for each $\mathcal{R}$. Assume there exists a $B \in \mathcal{C}$ such that $v(B) \neq v(B \setminus A)$. Looking at the chain $\emptyset \subset B\setminus A \subset B \subset I$ Lemma 5.5 yields the existnece of a measurable order $\mathcal{R}$ such that

$$(\varphi^{\mathcal{R}}v)(B \cap A) = (\varphi^{\mathcal{R}}v)(B \setminus (B \setminus A)) = v(B) - v(B \setminus A) \neq 0 \ ;$$

hence $(\varphi^{\mathcal{R}}v)(B \cap A) \neq 0$, which contradicts the fact that $|\varphi^{\mathcal{R}}v|(A) = 0$.

$\Longrightarrow$: Let $v \underset{w}{<} \mu$ where $\mu \in NA^+$. Let A be v-null and $\mathcal{R}$ be a measurable order. We shall show $|\varphi^{\mathcal{R}}v|(A) = 0$.

Define a measure $\mu_A(S) = \mu(S\setminus A)$; then $\mu_A \in NA^+$. We shall show that $v \underset{w}{<} \mu_A$. Let $U \subset T$ be such that $\mu_A(T\setminus U) = 0$, which means $\mu((T\setminus U)\setminus A) = 0$. Since $v \underset{w}{<} \mu$, it follows that $(T\setminus U)\setminus A$ is v-null. Remembering that A is v-null, we obtain

$$|v(T) - v(U)| = |(v(T) - v(T\setminus A)| + |v(T\setminus A) - v(U\setminus A)| + |v(U\setminus A) - v(U)|$$

$$= |v(T\setminus A) - v((T\setminus A)\setminus((T\setminus U)\setminus A))| = 0 .$$

Now, $v \underset{w}{<} \mu_A$, $\mu_A \in NA^+$ and Proposition 5.9 imply that $\varphi^{\mathcal{R}} v \underset{w}{<} \mu_A$ for each meausrable order $\mathcal{R}$. Now

$$\mu_A(A) = \mu(A\setminus A) = 0 \quad ;$$

therefore $(\varphi^{\mathcal{R}} v)(A) = 0$. Q.E.D.

The following Propositions (5.11, 5.13 and 5.14) will show properties of set functions in ORD (which are not necessarily non-atomic) which are inherited to the $\varphi^{\mathcal{R}} v$'s.

<u>Propositon 5.11</u>. Let $v \in ORD$, $\mu \in M^+$ and $v \underset{w}{<} \mu$. Then, $\{s\}$ is v-null v-null $\Longleftarrow$ $(\varphi^{\mathcal{R}} v)\{s\} = 0$ for each measurable order $\mathcal{R}$.

<u>Remark</u>: This proposition gives a sufficient condition that an element of I is not an atom for any $\varphi^{\mathcal{R}} v$, (where $\mathcal{R}$ are measurable orders). From this point of view this proposition is analoguous to Proposition 5.6.

Proof. The proof is similar to what we have done in the proof of Proposition 5.6; the trivial direction is the same, the other is similarly divided into three cases.

Cases 1 and 2: The proof is just as in Proposition 5.6, except that the last sentence of the second case should be replaced by: "we used 5.7 and the fact that $\{s\}$ is v-null".

Case 3: Let q_i^1, J be as in the proof of the third case in Proposition 5.6. We easily get that

$$(5.12) \qquad (\varphi^{\mathcal{A}} v)(\{s\}) = (\varphi^{\mathcal{A}} v)(J_1) - (\varphi^{\mathcal{A}} v)\ (I(s, \mathcal{A}))$$

$$= \lim_{i \to \infty} v(I(q_i^1, \mathcal{A})) - v(I(s, \mathcal{A})) \quad .$$

Now, choose any sequence - $\{a_i,\ i \geq 1\}$ consisting of elements of I such that $\mu(\{a_i\}) = 0$ for all $i \geq 1$, and $\{a_i | i \geq 1\} \cap \{q_i^1 | i \geq 1\} = \phi$. Clearly there exists such a sequence. Define an order $\mathcal{A}^*$ that puts the sequence a_i just after s and $a_i \underset{\mathcal{A}^*}{<} a_j$ iff $i < j$, that means

$$x \underset{\mathcal{A}^*}{\leq} y \iff \begin{cases} x, y \notin \{a_i\} \quad \text{and} \quad x \underset{\mathcal{A}}{\leq} y \ , \\ x \in \{a_i\}\ ,\ y \notin \{a_i\} \quad \text{and} \quad y \underset{\mathcal{A}}{\leq} s\ , \\ y \in \{a_i\},\ x \notin \{a_i\} \quad \text{and} \quad x \underset{\mathcal{A}}{\leq} s \\ x = a_i,\ x = a_j \quad \text{and} \quad i < j\ . \end{cases}$$

Note that $\mathcal{R}^*$ is a measurable order. Since $\mu(\{a_i\}) = 0$ for all i we get that $\{a_i, i \geq 1\}$ is μ-null, hence it is v-null and therefore

$$t \notin \{a_i; i \geq 1\} \Rightarrow v(I(t, \mathcal{R})) = v(I(t, \mathcal{R}^*)) \quad .$$

This holds in particular for $t = q_i^1$ $(i \geq 1)$. Now note that for all $i \geq 1$ $v(I(s, \mathcal{R})) = v(I(a_i, \mathcal{R}^*))$, hence we get (using 5.12) that

$$(\varphi^{\mathcal{R}} v)(\{s\}) = \lim_{i \to \infty} v(I(q_i^1, \mathcal{R})) - v(I(s, \mathcal{R}))$$

$$= \lim_{i \to \infty} v(I(q_i^1, \mathcal{R}^*)) - \lim_{i \to \infty} v(I(a_i, \mathcal{R}^*))$$

$$= (\varphi^{\mathcal{R}^*} v)(J_1 \cup \{a_i | i \geq 1\}) - \lim_{i \to \infty} (\varphi^{\mathcal{R}^*} v)\ (I(a_i, \mathcal{R}^*))$$

$$= (\varphi^{\mathcal{R}^*} v)(J_1 \cup \{a_i | i \geq 1\}) - (\varphi^{\mathcal{R}^*} v)(J_1 \cup \{a_i | i \geq 1\} = 0 \ .$$

Q.E.D.

Remark: Note that under the assumptions of Proposition 5.10, we could not repeat the proof of the third case in Proposition 5.6 since we could not know whether b is v-null or not. On the other hand we could not use there the trick used here, since the non-atomicity of v does not necessarily imply the existence of a denumerable v-null set.

Corollary 5.13: Let $v \in ORD$, $\mu \in M^+$ and $v \underset{w}{\leq} \mu$. Let A be a denumerable set in $\mathcal{C}$. Then $\mu(A) = 0 \implies \varphi^{\mathcal{R}}v(A) = 0$ for all measurable orders $\mathcal{R}$.

Proof. $\mu(A) = 0$ implies $\mu(\{s\}) = 0$ for all s in A. Hence all s in A are v-null. Now by Proposition 5.11 we get $(\varphi^{\mathcal{R}}v)(\{s\}) = 0$ for all s in A which completes our proof. Q.E.D.

Proposition 5.14. Let $v \in ORD$, $\mu \in M^+$. Then $v \underset{w}{<} \mu \iff \varphi^{\mathcal{R}}v \underset{w}{<} \mu$ for each measurable order $\mathcal{R}$.

Proof: Follow the proof of Proposition 5.9 and note that $|\varphi^{\mathcal{R}}v|(A) = 0$ (resp. $|\varphi^{\mathcal{R}^*}v|(A^*) = 0$) is not because v is non-atomic but because of the fact that $\mu(A) = 0$ (resp. $\mu(A^*) = 0$) and that A (resp. A^*) are denumerable, (use Corollary 5.13). Q.E.D.

Proposition 5.15. Let $v \in ORD$, $\mu \in M^+$ and $v \underset{w}{<} \mu$. Then A is v-null $\iff$ A is $\varphi^{\mathcal{R}}v$ null for all measurable orders $\mathcal{R}$.

Proof. Just as the proof of 5.10 except that $\mu_A \in M^+$ (instead of $\mu_A \in NA^+$). Q.E.D.

Remark. As mentioned in the remark preceding the proof of Proposition 5.10 there is an conjecture that every non-atomic set function in ORD is weakly continuous w.r.t some measure in NA^+. The analoguous conjecture in the general case would be that every set function in ORD is weakly continuous w.r.t some measure in M^+. If this is true, Proposition 5.15 would become: "Let $v \in ORD$. Then A is v-null $\iff$ A is $\varphi^{\mathcal{R}}v$ null for all measurable orders $\mathcal{R}$".

6. CHARACTERIZATION OF AC IN ORD

For $v \in \mathrm{ORD}$ write $K_v = \{\varphi^{\mathfrak{a}} v \mid$ is a measurable order$\}$ and $K'_v = \{|\varphi^{\mathfrak{a}} v| \mid \varphi^{\mathfrak{a}} v \in K_v\}$.

Proposition 6.1. Let v ORD be non-atomic. Then $v \in \mathrm{AC}$ iff K_v, or equivalently K'_v, are weakly sequentially compact, henceforth abbreviated w.s.c.)

Remark. A set S is w.s.c. in a Banach space X iff it is w.s.c. in any fixed closed linear subspace of X that contains S. One direction is immediate, the other follows from Hahn-Banach Theorem. Hence we do not have to mention in which space K_v and K'_v are w.s.c.

Lemma 6.2. A set K of σ-additive measures is w.s.c. iff $K' = \{|\mu| \,|\, \mu \in K\}$ is w.s.c.

Proof. Recall that "measure" in this section means "σ-additive totally finite signed measure". A necessary and sufficient condition for weak sequential compactness of a set K of measures [Dun-s, Theorem IV.9.2, p. 306] is that K is bounded and that there exists a positive measure λ such that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that: $\lambda(E) \leq \varepsilon$ for $E \in \mathcal{C}$ implies $\mu(E) \leq \varepsilon$ for all $\mu \in K$. (The last condition will be denoted: $\mu(E) \underset{\lambda(E)\to 0}{\to} 0$ uniformly w.r.t. μ in K.) Clearly the boundedness condition is equivalent in K and K' (since $\|\mu\| = \||\mu|\|$). Noting that

$|\mu(E)| \leq |\mu|(E)$ for all $E \in \mathcal{C}$ completes the proof that if K' is w.s.c. then K is w.s.c.

Finally let K be w.s.c. and λ the positive measure in the above condition.

For $\varepsilon > 0$, let δ be such that

$$\lambda(E) \leq \delta \implies |\mu(E)| \leq \frac{\varepsilon}{2} \qquad \forall \mu \in K$$

since λ is positive, we get for all $F \subset E$,

$$\lambda(E) \leq \delta \implies |\mu(F)| + |\mu(E \setminus F)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

hence for all $\mu \in K$

$$\lambda(E) \leq \delta \implies |\mu|(E) = \sup_{F \subset E} \{|\mu(F)| + \mu(E \setminus F)\} \leq \varepsilon \quad . \quad \text{Q.E.D.}$$

<u>Proof of Proposition 6.1</u>. Let K_v be w.s.c. then by [Dun-s, IV.9.2, p. 306] we know the existence of a positive measure λ such that $(\varphi^{\mathcal{R}} v)(E) \xrightarrow[\lambda(E) \to 0]{} 0$ uniformly w.r.t. $\varphi^{\mathcal{R}} v \in K_v$. If we follow the proof of that Theorem we will find that λ is defined as the sum of a series of modified members of the weakly sequentially compact set. As v is non-atomic and v ORD, each $\varphi^{\mathcal{R}} v$ is non-atomic (Proposition 5.6); hence all $\varphi^{\mathcal{R}} v$'s are non-atomic, and therefore we might demand λ being non-atomic. So let us assume $\lambda \in NA^+$.

We are going to show $v \ll \lambda$. If this assumption does not hold there exists $\varepsilon_0 > 0$ such that for any given integer $n \geq 1$ there is a chain

$$\emptyset \subset S_1^{(n)} \subset T_1^{(n)} \subset \cdots \subset S_{k_n}^{(n)} \subset T_{k_n}^{(n)} \subset I$$

such that

$$\sum_{i=1}^{k_n} |\lambda(T_i^{(n)}) - \lambda(S_i^{(n)})| \leq \frac{1}{n}$$

and

$$\sum_{i=1}^{k_n} |v(T_i^{(n)}) - v(S_i^{(n)})| \geq \varepsilon_0 \quad .$$

Clearly, by omitting some sets and changing indices, we may demand

$$\left| \sum_{i=1}^{k_n} (v(T_i^{(n)}) - v(S_i^{(n)})) \right| \geq \frac{\varepsilon_0}{2} \quad .$$

Set

$$A_n = \bigcup_{i=1}^{n} (T_i^{(n)} \setminus S_i^{(n)}) \ .$$

Since λ is positive,

$$\lambda(A_n) = \sum_{i=1}^{k_n} |\lambda(T_i^{(n)}) - \lambda(S_i^{(n)})| \le \frac{1}{n}$$

Lemma 5.5 yields the existence of measurable orders $\mathcal{R}_n$ such that

$$(\varphi^{\mathcal{R}_n} v)(A_n) = (\varphi^{\mathcal{R}_n} v)\{\bigcup_{i=1}^{k_n} (T_i^{(n)} \setminus S_i^{(n)})\} = \sum_{i=1}^{k_n} (v(T_i^{(n)}) - v(S_i^{(n)}) \quad .$$

Then

$$|(\varphi^{\mathcal{R}_n} v)(A_n)| = |\sum_{i=1}^{k_n} (v(T_i^{(n)}) - v(S_i^{(n)}))| \ge \frac{\varepsilon_0}{2} \quad .$$

This, and the fact $\lambda(A_n) \le \frac{1}{n}$ contradicts the uniform limit $(\varphi^{\mathcal{R}} v)(E) \underset{\lambda(E)\to 0}{\to} 0$ with respect to $\varphi^{\mathcal{R}} v \in K_v$; hence $v << \lambda$.

Now let $v \in AC$. To prove that K_v is w.s.c. it is sufficient to show that K_v is bounded and that there exists a measure $\lambda \in M^+$ such that $(\varphi^{\mathcal{R}} v)(E) \underset{\lambda(E)\to 0}{\to} 0$ uniformly w.r.t. $\varphi^{\mathcal{R}} v \in K_v$. K_v is clearly bounded since $\|\varphi^{\mathcal{R}} v\| \le \|v\|$ for all measurable orders (Proposition 5.2). To prove the uniform limit let λ be the measure in NA^+ such that $v << \lambda$, let $\varepsilon > 0$ be given and let $\delta > 0$ correspond to $\varepsilon/2$ in accordance with the definition $v << \lambda$. Let be a measurable order and let $H(\mathcal{R})$ be the field (not σ-field) generated by the initial segments. Clearly a set in $H(\mathcal{R})$ is of the form $U = \bigcup_{i=1}^{n} [s_i, t_i)_{\mathcal{R}}$, where

$$s_1 <_{\mathcal{R}} t_1 <_{\mathcal{R}} s_2 <_{\mathcal{R}} \cdots <_{\mathcal{R}} s_n <_{\mathcal{R}} t_n$$

Look at the subchain Λ consisiting of the links $\{I(s_i, \mathcal{R}), I(t_i, \mathcal{R})\}$ then

$$\|\lambda\|_\Lambda = \sum_{i=1}^{n} |\lambda(I(t_i, \mathcal{R})) - \lambda(I(s_i, \mathcal{R}))| = \lambda(U)$$

and

$$\|v\|_\Lambda = \sum_{i=1}^{n} |v(I(t_i, \mathcal{R})) - v(I(s_i, \mathcal{R}))|$$

$$\geq \left|\sum_{i=1}^{n} (v(I(t_i, \mathcal{R})) - v(I(s_i, \mathcal{R})))\right| = |(\varphi^{\mathcal{R}}v)(U)| \; .$$

Hence

(6.3) $\qquad \lambda(U) \leq \delta \implies \|\lambda\|_\Lambda \leq \delta \implies \|v\|_\Lambda \leq \frac{\varepsilon}{2} \implies |(\varphi^{\mathcal{R}}v)(U)| \leq \frac{\varepsilon}{2}$.

Now by a standard approximation Theorem*, every meaeurable set can be approximated by members of $H(\mathcal{R})$ simulatneously w.r.t. μ and w.r.t $|\varphi^{\mathcal{R}}v|$. Hence if $\mu(S) < \delta$ there exists a $U \in H(\mathcal{R})$ such that $\mu(U) \leq \delta$ and $|\varphi^{\mathcal{R}}v|(U \nabla S) \leq \frac{\varepsilon}{2}$. This and 6.3 imply that if $\mu(S) < \delta$ then $|(\varphi^{\mathcal{R}}v)(S)| \leq \varepsilon$. Hence

$$\mu(S) \leq \frac{\delta}{2} \implies |(\varphi^{\mathcal{R}}v)(S)| \leq \varepsilon$$

for all measurable orders $\mathcal{R}$. This completes the proof that K_v is w.s.c. Q.E.D.

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*One uses [H, p. 56, Theorem D] on the measure $\mu + |\varphi^{\mathcal{R}}v|$.

Proposition 6.4. Let $v \in ORD$. Then there exists a measure λ such that $v \ll \lambda$ iff K_v (or equivalently K'_v) i w.s.c.

Proof. Exactly as the proof of Proposition 6.1 replacing $\lambda \in NA^+$ by $\lambda \in M^+$ and omitting the part which proves the non-atomicity of λ.

Q.E.D.

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